

On explicit numerical schemes for the CIR process

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Abstract

In this paper we generalize an explicit numerical scheme for the CIR process that we have proposed before. The advantage of the new proposed scheme is that preserves positivity and is well posed for a (little bit) broader set of parameters among the positivity preserving schemes. The order of convergence is at least logarithmic in general and for a smaller set of parameters is at least $1/4$. Next we give a different explicit numerical scheme based on exact simulation and we use this idea to approximate the two factor CIR model. Finally, we give a second explicit numerical scheme for the two factor CIR model based on the idea of the second section.

Keywords: Explicit numerical scheme, CIR process, positivity preserving, order of convergence.

AMS subject classification: 60H10, 60H35.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a complete probability space with a filtration and let a Wiener process $(W_t)_{t \geq 0}$ defined on this space. We consider here the CIR process, (see [7]),

$$x_t = x_0 + \int_0^t (kl - kx_s)ds + \sigma \int_0^t \sqrt{x_s}dW_s, \quad (1)$$

where $k, l, \sigma \geq 0$. It is well known that this sde has a unique strong solution which remain nonnegative. This stochastic process is widely used in financial mathematics. It is well known that one can use exact simulation methods to construct the true solution but the drawback of such an approach is the computational time that requires. Therefore, many researchers work on construction of fast and efficient methods to approximate this process. In [14] the authors proposed a modified Euler scheme for the approximation of the CIR process. However, this scheme does not preserve positivity which is a desirable property in some cases. Next, in [1], the author proposes a positivity preserving numerical scheme which is strongly convergent but not for all possible parameters. In [11] we have proposed another positivity preserving numerical scheme for the CIR process and our goal here is to propose a generalization of this scheme in order to be well posed for a broader class of parameters being of course positivity preserving scheme.

Let $0 = t_0 < t_1 < \dots < t_n = T$ and set $\Delta = \frac{T}{n}$. Consider the following stochastic process

$$y_t = \left(\frac{\sigma}{2(1 + ka\Delta)}(W_t - W_{t_k}) + \sqrt{y_{t_k}(1 - \frac{k\Delta}{1 + ka\Delta}) + \frac{\Delta}{1 + ka\Delta}(kl - \frac{\sigma^2}{4(1 + ka\Delta)})} \right)^2 = (z_t)^2, \quad (2)$$

for $t \in (t_k, t_{k+1}]$ and a parameter $a \in [0, 1]$ where

$$z_t = \frac{\sigma}{2(1 + ka\Delta)}(W_t - W_{t_k}) + \sqrt{y_{t_k}(1 - \frac{k\Delta}{1 + ka\Delta}) + \frac{\Delta}{1 + ka\Delta}(kl - \frac{\sigma^2}{4(1 + ka\Delta)})},$$

for $t \in (t_k, t_{k+1}]$.

Note that this process is well defined when $a\Delta \geq \frac{\sigma^2 - 4kl}{4k^2l}$ and has the differential form, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} y_t = & y_{t_k} + \Delta \left(kl - \frac{\sigma^2}{4(1+ka\Delta)} - k(1-a)y_{t_k} - kay_t \right) + \int_{t_k}^t \frac{\sigma^2}{4(1+ka\Delta)} ds \\ & + \sigma \int_{t_k}^t \text{sgn}(z_s) \sqrt{y_s} dW_s. \end{aligned} \quad (3)$$

To obtain the above form we first use Ito's formula on y_t and then some simple rearrangements. This stochastic process is not continuous in all $[0, T]$, because there are jumps at the nodes t_k .

The numerical scheme that we propose here to approximate the CIR process is the following,

$$y_{t_{k+1}} = \left(\frac{\sigma}{2(1+ka\Delta)} (W_{t_{k+1}} - W_{t_k}) + \sqrt{y_{t_k} \left(1 - \frac{k\Delta}{1+ka\Delta} \right) + \frac{\Delta}{1+ka\Delta} \left(kl - \frac{\sigma^2}{4(1+ka\Delta)} \right)} \right)^2,$$

with $y_{t_0} = x_0$. Using Ito's formula one can easily see that y_t in (2) is the unique solution of the stochastic differential equation (3). Therefore it is clear that is positivity preserving and well defined for $a\Delta \geq \frac{\sigma^2 - 4kl}{4k^2l}$. This set of parameters is (a little bit) broader than the existing numerical schemes that preserves positivity, which usually is $4kl \geq \sigma^2$. The main goal of future research will be the construction of positivity preserving numerical methods that will be well posed for all possible parameters, see for example [4] for such a method without a theoretical convergence result.

For generalizations of the semi discrete method see [12], [13].

2 Main Results

We will use a compact form of (3), for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} y_t = & x_0 + \int_0^t (kl - k(1-a)y_{\tilde{s}} - kay_{\tilde{s}}) ds \\ & + \int_t^{t_{k+1}} \left(kl - \frac{\sigma^2}{4(1+ka\Delta)} - k(1-a)y_{t_k} - kay_t \right) ds + \sigma \int_0^t \text{sgn}(z_s) \sqrt{y_s} dW_s, \end{aligned}$$

where

$$\tilde{s} = \begin{cases} t_{j+1}, & \text{when } s \in [t_j, t_{j+1}], \quad j = 0, \dots, k-1 \\ t, & \text{when } s \in (t_k, t] \end{cases}$$

and $\hat{s} = t_j$ when $s \in (t_j, t_{j+1}]$, $j = 0, \dots, k$. Therefore, y_t remains nonnegative as is the same as in (3).

We will remove the term $\text{sgn}(z_s)$ by changing the Brownian motion. Set

$$\hat{W}_t = \int_0^t \text{sgn}(z_s) dW_s.$$

It is easy to see that \hat{W} is a continuous martingale on \mathcal{F}_t with variation $\langle \hat{W}, \hat{W} \rangle = t$. Therefore, using Levy's martingale characterization of Brownian motion (see [16], p. 157) we deduce that \hat{W}_t is also a Brownian motion. Therefore, y_t satisfies the following equation,

$$\begin{aligned} y_t = & x_0 + \int_0^t (kl - k(1-a)y_{\hat{s}} - kay_{\hat{s}}) ds \\ & + \int_t^{t_{k+1}} \left(kl - \frac{\sigma^2}{4(1+ka\Delta)} - k(1-a)y_{t_k} - kay_t \right) ds + \sigma \int_0^t \sqrt{y_s} d\hat{W}_s, \end{aligned}$$

Let now the following sde,

$$\hat{x}_t = x_0 + \int_0^t (kl - k\hat{x}_s)ds + \sigma \int_0^t \sqrt{\hat{x}_s}d\hat{W}_s, \quad (4)$$

where \hat{W}_t , constructed as above, is a Brownian motion depending on Δ . For each Δ the above problem has a unique solution which has the same transition density (see [10], p. 122), independent of Δ . We will show that $\mathbb{E}|\hat{x}_t - y_t|^2 \rightarrow 0$ as $\Delta \rightarrow 0$ and therefore our approximation converges in the mean square sense to a stochastic process that is equal in distribution to the unique solution of (1). We will denote \hat{W}, \hat{x} again by W, x for notation simplicity.

Assumption A We suppose that $x_0 \geq 0$ a.s. $\mathbb{E}x_0^p < A$ for some $p \geq 2$, $d = kl - \frac{\sigma^2}{4(1+ka\Delta)} \geq 0$ and $\Delta(1-a) \leq \frac{1}{k}$.

Lemma 1 (Moment bounds) *Under Assumption A we have the moment bounds,*

$$\mathbb{E}y_t^p + \mathbb{E}x_t^p < C,$$

for some $C > 0$

Proof. Note that

$$0 \leq y_t \leq v_t = x_0 + Tkl + \sigma \int_0^t \sqrt{y_s}dW_s.$$

Consider the stopping time $\theta_R = \inf\{t \geq 0 : v_t > R\}$. Using Ito's formula on $v_{t \wedge \theta_R}^p$ we obtain,

$$v_{t \wedge \theta_R}^p = (x_0 + Tkl)^p + \frac{p(p-1)}{2} \sigma^2 \int_0^t v_{s \wedge \theta_R}^{p-2} y_{s \wedge \theta_R} ds + p\sigma \int_0^t v_{s \wedge \theta_R}^{p-1} \sqrt{y_{s \wedge \theta_R}} dW_s.$$

Taking expectations on both sides and noting that $y_t \leq v_t$, we arrive at

$$\begin{aligned} \mathbb{E}v_{t \wedge \theta_R}^p &\leq \mathbb{E}(x_0 + Tkl)^p + \frac{p(p-1)}{2} \sigma^2 \int_0^t \mathbb{E}v_{s \wedge \theta_R}^{p-1} ds \\ &\leq \mathbb{E}(x_0 + Tkl)^p + \frac{p(p-1)}{2} \sigma^2 \int_0^t (\mathbb{E}v_{s \wedge \theta_R}^p)^{\frac{p-1}{p}} ds \end{aligned}$$

Using now a Gronwall type theorem (see [19], Theorem 1, p. 360), we arrive at

$$\mathbb{E}v_{t \wedge \theta_R}^p \leq \left([\mathbb{E}(x_0 + Tkl)^p]^{\frac{p-1}{p}} + \frac{T}{2}(p-1)\sigma^2 \right)^{\frac{p}{p-1}}. \quad (5)$$

But $\mathbb{E}v_{t \wedge \theta_R}^p = \mathbb{E}(v_{t \wedge \theta_R}^p \mathbb{I}_{\{\theta_R \geq t\}}) + R^p P(\theta_R < t)$. That means that $P(t \wedge \theta_R < t) = P(\theta_R < t) \rightarrow 0$ as $R \rightarrow \infty$ so $t \wedge \theta_R \rightarrow t$ in probability and noting that θ_R increases as R increases we have that $t \wedge \theta_R \rightarrow t$ almost surely too, as $R \rightarrow \infty$. Going back to (4) and using Fatou's lemma we obtain,

$$\mathbb{E}v_t^p \leq \left([\mathbb{E}(x_0 + Tkl)^p]^{\frac{p-1}{p}} + \frac{T(p-1)\sigma^2}{2} \right)^{\frac{p}{p-1}}$$

We have assume in our assumptions that $\mathbb{E}x_0^p < \infty$ in order the term $\mathbb{E}(x_0 + Tkl)^p$ to be well posed.

The same holds for x_t (see for example [8]). \square

Consider the auxiliary stochastic process, for $t \in (t_k, t_{k+1}]$,

$$h_t = x_0 + \int_0^t (kl - k(1-a)y_{\tilde{s}} - kay_{\hat{s}})ds + \sigma \int_0^t \sqrt{y_s}dW_s, \quad (6)$$

where \tilde{s}, \hat{s} defined as before.

Lemma 2 *We have the following estimates,*

$$\begin{aligned}\mathbb{E}|h_s - y_s|^2 &\leq C_1 \Delta^2 \text{ for any } s \in [0, T] \\ \mathbb{E}|h_s - y_r|^2 &\leq C_2 \Delta \text{ when } s \in [t_k, t_{k+1}] \text{ and } r = t_k \text{ or } t_{k+1} \\ \mathbb{E}|h_s|^2 &< A, \text{ for any } s \in [0, T].\end{aligned}$$

Proof. Using the moment bound for y_t we easily obtain the fact that

$$\mathbb{E}|h_t - y_t|^2 \leq C\Delta^2.$$

Next, we have

$$\mathbb{E}|h_s - y_{t_k}|^2 \leq 2\mathbb{E}|h_s - y_s|^2 + 2\mathbb{E}|y_s - y_{t_k}|^2 \leq C\Delta^2 + C\Delta \leq C\Delta.$$

Moreover,

$$\mathbb{E}|h_s - y_{t_{k+1}}|^2 \leq \mathbb{E}|h_s - y_s|^2 + 2\mathbb{E}|y_s - y_{t_{k+1}}|^2 \leq C\Delta^2 + C\Delta \leq C\Delta.$$

Finally, to get the moment bound for h_t we just use the fact that is close to y_t , i.e.

$$\mathbb{E}h_t^2 \leq 2\mathbb{E}|h_t - y_t|^2 + 2\mathbb{E}y_t^2 \leq C.$$

□

Theorem 1 *If Assumption A holds then*

$$\mathbb{E}|x_t - y_t|^2 \leq C \frac{1}{\sqrt{\ln n}}$$

for any $t \in [0, T]$.

Proof.

Applying Ito's formula on $|x_t - h_t|^2$ we obtain

$$\begin{aligned}\mathbb{E}|x_t - h_t|^2 &\leq 2k(1-a) \int_0^t \mathbb{E}|x_s - h_s||y_{\hat{s}} - x_s|ds + 2ka \int_0^t \mathbb{E}|x_s - h_s||y_{\bar{s}} - x_s|ds \\ &\quad + \sigma^2 \int_0^t \mathbb{E}|x_s - y_s|ds\end{aligned}\tag{7}$$

Let us estimate the above quantities. It is easy to see that, for example,

$$\mathbb{E}|x_s - h_s||y_{\hat{s}} - x_s| \leq \mathbb{E}|x_s - h_s|(|x_s - h_s| + |h_s - y_{\hat{s}}|)$$

Therefore, we obtain, using Cauchy-Schwarz inequality

$$\begin{aligned}\mathbb{E}|x_s - h_s||y_{\hat{s}} - x_s| &\leq \mathbb{E}|x_s - h_s|^2 + \sqrt{\mathbb{E}|x_s - h_s|^2} \sqrt{\mathbb{E}|h_s - y_{\hat{s}}|^2}, \\ \mathbb{E}|x_s - h_s||y_{\bar{s}} - x_s| &\leq \mathbb{E}|x_s - h_s|^2 + \sqrt{\mathbb{E}|x_s - h_s|^2} \sqrt{\mathbb{E}|h_s - y_{\bar{s}}|^2}\end{aligned}$$

Summing up we arrive at

$$\mathbb{E}|x_t - h_t|^2 \leq C\sqrt{\Delta} + C \int_0^t \mathbb{E}|x_s - h_s|^2 ds + \int_0^t \mathbb{E}|x_s - h_s| ds.\tag{8}$$

Therefore, we have to estimate $\mathbb{E}|x_t - h_t|$. Let the non increasing sequence $\{e_m\}_{m \in \mathbb{N}}$ with $e_m = e^{-m(m+1)/2}$ and $e_0 = 1$. We introduce the following sequence of smooth approximations of $|x|$, (method of Yamada and Watanabe, [23])

$$\phi_m(x) = \int_0^{|x|} dy \int_0^y \psi_m(u) du,$$

where the existence of the continuous function $\psi_m(u)$ with $0 \leq \psi_m(u) \leq 2/(mu)$ and support in (e_m, e_{m-1}) is justified by $\int_{e_m}^{e_{m-1}} (du/u) = m$. The following relations hold for $\phi_m \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ with $\phi_m(0) = 0$,

$$|x| - e_{m-1} \leq \phi_m(x) \leq |x|, \quad |\phi'_m(x)| \leq 1, \quad x \in \mathbb{R},$$

$$|\phi''_m(x)| \leq \frac{2}{m|x|}, \quad \text{when } e_m < |x| < e_{m-1} \text{ and } |\phi''_m(x)| = 0 \text{ otherwise.}$$

Applying Ito's formula on $\phi_m(x_t - h_t)$ we obtain

$$\begin{aligned} \mathbb{E}\phi_m(x_t - h_t) &= \int_0^t \mathbb{E}\phi'_m(x_s - h_s)(k(1-a)(y_{\bar{s}} - x_s) + ka(y_{\bar{s}} - x_s))ds \\ &\quad + \int_0^t \frac{1}{2} \mathbb{E}\phi''_m(x_s - h_s)(\sigma\sqrt{y_s} - \sigma\sqrt{x_s})^2 ds. \end{aligned}$$

We continue by estimating

$$\begin{aligned} &\mathbb{E}\phi'_m(x_s - h_s)(k(1-a)(y_{\bar{s}} - x_s) + ka(y_{\bar{s}} - x_s)) \\ &\leq k\mathbb{E}|x_s - h_s| + k(1-a)\mathbb{E}|h_s - y_{\bar{s}}| + ka\mathbb{E}|h_s - y_{\bar{s}}| \\ &\leq k\mathbb{E}|x_s - h_s| + C\sqrt{\Delta}. \end{aligned}$$

Next,

$$\mathbb{E}\phi''_m(x_s - h_s)(\sigma\sqrt{y_s} - \sigma\sqrt{x_s})^2 \leq \frac{4\sigma^2}{m} + \frac{4\sigma^2}{m} \mathbb{E} \frac{|h_s - y_s|}{e_m}$$

Working as before and using Lemma 2 we get

$$\mathbb{E}\phi''_m(x_s - h_s)(\sigma\sqrt{y_s} - \sigma\sqrt{x_s})^2 \leq \frac{4\sigma^2}{m} + C \frac{\sqrt{\Delta}}{me_m} + C \frac{\sqrt{\Delta}}{m}.$$

Therefore,

$$\mathbb{E}|x_t - h_t| \leq e_{m-1} + \frac{4\sigma^2 T}{m} + C \frac{\sqrt{\Delta}}{me_m} + C \frac{\sqrt{\Delta}}{m} + k \int_0^t \mathbb{E}|x_s - h_s| ds.$$

Use now Gronwall's inequality and substitute in (9) and then again Gronwall's inequality we arrive at

$$\mathbb{E}|x_t - h_t|^2 \leq C\sqrt{\Delta} + C \frac{\sqrt{\Delta}}{me_m} + e_{m-1}.$$

Choosing $m = \sqrt{\ln n^{\frac{1}{3}}}$ we deduce that

$$\mathbb{E}|x_t - h_t|^2 \leq C \frac{1}{\sqrt{\ln n}}$$

But

$$\mathbb{E}|x_t - y_t|^2 \leq 2\mathbb{E}|x_t - h_t|^2 + 2\mathbb{E}|h_t - y_t|^2 \leq C \frac{1}{\sqrt{\ln n}}$$

□

3 On the polynomial rate of convergence

We study in this section the polynomial order of convergence of our scheme. We use a stochastic time change proposed in [5]. For simplicity, we take $a = 0$.

Our result is as follows.

Proposition 1 *If*

$$\sigma^2 \leq 2kl \text{ and } \frac{1}{16}(\frac{2kl}{\sigma^2} - 1)^2 > 1$$

the following rate of convergence holds, assuming that $x_0 \in \mathbb{R}$ and $x_0 > 0$,

$$\mathbb{E}|x_t - y_t|^2 \leq C\Delta.$$

Proof.

Define the process

$$\gamma(t) = \int_0^t \frac{ds}{(\sqrt{x_s} + \sqrt{h_s})^2},$$

and then the stopping time defined by

$$\tau_l = \inf\{s \in [0, T] : 2\sigma^2\gamma(s) + 3ks \geq l\}.$$

Using Ito's formula on $|x_\tau - h_\tau|^2$ with τ a stopping time, we obtain

$$\begin{aligned} \mathbb{E}(x_\tau - h_\tau)^2 &\leq \int_0^\tau (2k\mathbb{E}|x_s - h_s||y_s - x_s| + \sigma^2\mathbb{E}|\sqrt{y_s} - \sqrt{x_s}|^2) ds \\ &\leq \int_0^\tau (2k\mathbb{E}|x_s - h_s|^2 + 2k\mathbb{E}|x_s - h_s||h_s - y_s| + \sigma^2\mathbb{E}|\sqrt{y_s} - \sqrt{x_s}|^2) ds \\ &\leq 3k \int_0^\tau \mathbb{E}|x_s - h_s|^2 ds + \sigma^2 \int_0^\tau \mathbb{E}|\sqrt{y_s} - \sqrt{x_s}|^2 ds + C\Delta. \end{aligned}$$

Now we work on

$$\int_0^\tau \mathbb{E}|\sqrt{y_s} - \sqrt{x_s}|^2 ds \leq \int_0^\tau 2\mathbb{E}|\sqrt{x_s} - \sqrt{h_s}|^2 + 2\mathbb{E}|\sqrt{h_s} - \sqrt{y_s}|^2 ds \leq 2 \int_0^\tau \mathbb{E}|\sqrt{x_s} - \sqrt{h_s}|^2 ds + C\Delta.$$

But

$$\int_0^\tau \mathbb{E}|\sqrt{x_s} - \sqrt{h_s}|^2 ds = \int_0^\tau \mathbb{E} \frac{|x_s - h_s|^2}{(\sqrt{x_s} + \sqrt{h_s})^2} ds.$$

Therefore,

$$\mathbb{E}(x_\tau - h_\tau)^2 \leq C\Delta + \mathbb{E} \int_0^\tau |x_s - h_s|^2 (3ks + 2\sigma^2\gamma_s)' ds. \quad (9)$$

Now, for $\tau = \tau_l$, we use the change of variables setting $u = 3ks + 2\sigma^2\gamma_s$ and therefore $s = \tau_u$ obtaining,

$$\mathbb{E}(x_{\tau_l} - h_{\tau_l})^2 \leq C\Delta + \int_0^l \mathbb{E}|x_{\tau_u} - h_{\tau_u}|^2 du.$$

Using Gronwall's inequality we obtain,

$$\mathbb{E}|x_{\tau_l} - h_{\tau_l}|^2 \leq Ce^l \Delta. \quad (10)$$

Going back to (9), for $\tau = t \in [0, T]$, we have under the change of variables $u = 2\sigma^2\gamma_s + 3ks$,

$$\begin{aligned}\mathbb{E}(x_t - h_t)^2 &\leq C\Delta + \mathbb{E} \int_0^{3kT+2\sigma^2\gamma_T} |x_{\tau_u} - h_{\tau_u}|^2 du \\ &\leq C\Delta + \int_0^\infty \mathbb{E} \left(\mathbb{I}_{\{3kT+2\sigma^2\gamma_T \geq u\}} |x_{\tau_u} - h_{\tau_u}|^2 \right) du.\end{aligned}\tag{11}$$

Noting that

$$\begin{aligned}&\int_0^\infty \mathbb{E} \left(\mathbb{I}_{\{3kT+2\sigma^2\gamma_T \geq u\}} |x_{\tau_u} - h_{\tau_u}|^2 \right) du \\ &\leq \int_0^{3kT} \mathbb{E} |x_{\tau_u} - h_{\tau_u}|^2 du + \int_{3kT}^\infty \mathbb{P}(3kT + 2\sigma^2\gamma_T \geq u) \mathbb{E} (|x_{\tau_u} - h_{\tau_u}|^2 \mid \{3kT + 2\sigma^2\gamma_T \geq u\}) du, \\ &\leq C\Delta + \int_0^\infty \mathbb{P}(2\sigma^2\gamma_T \geq u) \mathbb{E} |x_{\tau_u} - h_{\tau_u}|^2 du\end{aligned}$$

and then we arrive using (10)

$$\mathbb{E}(x_t - h_t)^2 \leq C\Delta \left(1 + C \int_0^\infty \mathbb{P}(2\sigma^2\gamma_T \geq u) e^u du \right).$$

We will estimate now the following,

$$\mathbb{P}(2\sigma^2\gamma_T \geq u) \leq \frac{1}{e^{mu}} \mathbb{E}(e^{2\sigma^2 m \gamma_T})$$

Choose $m = \frac{1}{16}(\frac{2kl}{\sigma^2} - 1)^2$ and use Thm. 3.1 of [17] to end the proof. \square

In order to avoid the difficulties from the appearance of the term $\text{sgn}(z_t)$ we have changed the Brownian motion. Below, we give a lemma which one can use to prove strong convergence without changing the Brownian motion and the difference is that the order of convergence is, at least, $\Delta^{1/4-\varepsilon}$ for any $\varepsilon > 0$. We prove it for the case $a = 0$ for simplicity but the same result holds for any $a \in [0, 1]$.

Lemma 3 *We have the following estimate,*

$$\mathbb{E} y_t (\text{sgn}(z_t) - 1)^2 \leq C\Delta^{\frac{1}{2}-\varepsilon},$$

for any $\varepsilon > 0$.

Proof. We begin with, when $t \in [t_k, t_{k+1}]$,

$$\begin{aligned}\mathbb{E} y_t (\text{sgn}(z_t) - 1)^2 &= 4\mathbb{E} y_t \mathbb{I}_{\{z_t \leq 0\}} \leq 4\mathbb{E} |y_t - y_{t_k}| + 4\mathbb{E} y_{t_k} \mathbb{I}_{\{z_t \leq 0\}} \\ &\leq C\Delta + 4\mathbb{E} y_{t_k} \mathbb{I}_{\{z_t \leq 0\}} \mathbb{I}_{\{y_{t_k} \leq \Delta^{1/2-\varepsilon}\}} + 4\mathbb{E} y_{t_k} \mathbb{I}_{\{z_t \leq 0\}} \mathbb{I}_{\{y_{t_k} > \Delta^{1/2-\varepsilon}\}} \\ &\leq C\Delta^{1/2-\varepsilon} + 4\mathbb{E} y_{t_k} \mathbb{I}_{\{z_t \leq 0\} \cap \{y_{t_k} > \Delta^{1/2-\varepsilon}\}}\end{aligned}$$

We have used Lemma 2 to obtain the third inequality, estimating the term $\mathbb{E} |y_t - y_{t_k}|$. But

$$\begin{aligned}\{z_t \leq 0\} \cap \{y_{t_k} > \Delta^{1/2-\varepsilon}\} &= \left\{ W_t - W_{t_k} \leq -\frac{2}{\sigma} \sqrt{y_{t_k}(1-k\Delta) + \Delta(kl - \frac{\sigma^2}{4})} \right\} \cap \{y_{t_k} > \Delta^{1/2-\varepsilon}\} \\ &\subseteq \left\{ W_t - W_{t_k} \leq -\frac{2}{\sigma} \sqrt{1-k\Delta} \sqrt{\Delta^{1/2-\varepsilon}} \right\}.\end{aligned}$$

Since the increment $W_t - W_{t_k}$ is normally distributed with mean zero and variance $t - t_k$ we have that

$$\mathbb{P} \left(\{z_t \leq 0\} \cap \{y_{t_k} > \Delta^{1/2-\varepsilon}\} \right) \leq \frac{\sqrt{t-t_k}}{\sqrt{2\pi(t-t_k)}} \int_{\frac{2\sqrt{1-k\Delta}\sqrt{\Delta^{1/2-\varepsilon}}}{\sqrt{t-t_k}}}^\infty e^{-y^2/2} dy \leq \frac{C\Delta^\varepsilon}{e^{C/\Delta^\varepsilon}}.$$

We have used the inequality of problem 9.22, p.112 of [16] to obtain the last inequality. Now we have, using the moment bounds for the numerical solution,

$$\mathbb{E}y_{t_k} \mathbb{I}_{\{z_t \leq 0\} \cap \{y_{t_k} > \Delta^{1/2-\varepsilon}\}} \leq C\mathbb{P}\left(\{z_t \leq 0\} \cap \{y_{t_k} > \Delta^{1/2-\varepsilon}\}\right)$$

Noting that $\frac{\Delta^\varepsilon}{e^{1/\Delta^\varepsilon}} \rightarrow 0$ faster than any power of Δ we have that

$$\mathbb{E}y_t(\text{sgn}(z_t) - 1)^2 \leq C\Delta^{\frac{1}{2}-\varepsilon}.$$

□

4 An explicit scheme for the CIR process using exact simulation

Consider the following equation.

$$x_t = x_0 + \int_0^t (kl - kx_s)ds + \int_0^t \sigma \sqrt{x_s} dW_s.$$

Our starting point is the exact simulation for the CIR process for some specific parameters. If $d = \frac{4kl}{\sigma^2} \in \mathbb{N}$ then we can simulate this process exactly (see [10], p. 133). Indeed, the exact simulation is given by

$$r(t_{i+1}) = \sum_{j=1}^d \left(e^{-\frac{1}{2}k\Delta} \sqrt{\frac{r(t_i)}{d}} + \frac{\sigma}{2} \sqrt{\frac{1}{k}(-e^{-k\Delta})} Z_{i+1}^{(j)} \right)^2,$$

where $(Z_i^{(1)}, \dots, Z_i^{(d)})$ are standard normal d -vectors, independent for different values of i . Therefore, the idea (see [11]) is to split a part of the drift term and the remaining drift coefficient will be such that we can simulate it exactly. Then, we will study the error produced by this splitting. First we assume that $d > 1$ and we will propose an explicit numerical scheme that preserves positivity and converges in the mean square sense with, at least, logarithmic order. For the case $2kl > 5\sigma^2$ we will show that this solution converges in the mean square sense with $1/2$ order of convergence.

4.1 The general case $d > 1$

We will use the main idea of [11] and propose the following semi discrete numerical scheme,

$$y_t = y_{t_k} + \Delta k_1 l - y_{t_k} \Delta k_1 + \int_{t_k}^t (k_2 l - k_2 y_s) ds + \sigma \int_{t_k}^t \sqrt{y_s} dW_s,$$

where $k = k_1 + k_2$ and $\frac{4k_2 l}{\sigma^2} = [\frac{4kl}{\sigma^2}]$ and by $[\cdot]$ we denote the integer part. The above sde has a unique strong solution which can be simulated exactly and is well posed when $\Delta < \frac{1}{k_1}$. A compact form of the numerical scheme is,

$$y_t = x_0 + \int_0^t (kl - k_2 y_s - k_1 y_{\hat{s}}) ds + \int_t^{t_{k+1}} (k_1 l - k_1 y_{\hat{s}}) ds + \sigma \int_0^t \sqrt{y_s} dW_s, \quad t \in (t_k, t_{k+1}].$$

Lemma 4 (Moment bounds) *Under Assumption A we have the moment bounds,*

$$\mathbb{E}y_t^p + \mathbb{E}x_t^p < C,$$

for some $C > 0$

Proof. Note that

$$0 \leq y_t \leq v_t = x_0 + Tkl + \sigma \int_0^t \sqrt{y_s} dW_s.$$

Consider the stopping time $\theta_R = \inf\{t \geq 0 : v_t > R\}$. Using Ito's formula on $v_{t \wedge \theta_R}^p$ we obtain,

$$v_{t \wedge \theta_R}^p = (x_0 + Tkl)^p + \frac{p(p-1)}{2} \sigma^2 \int_0^t v_{s \wedge \theta_R}^{p-2} y_{s \wedge \theta_R} ds + p\sigma \int_0^t v_{s \wedge \theta_R}^{p-1} \sqrt{y_{s \wedge \theta_R}} dW_s.$$

Taking expectations on both sides and noting that $y_t \leq v_t$, we arrive at

$$\begin{aligned} \mathbb{E}v_{t \wedge \theta_R}^p &\leq \mathbb{E}(x_0 + Tkl)^p + \frac{p(p-1)}{2} \sigma^2 \int_0^t \mathbb{E}v_{s \wedge \theta_R}^{p-1} ds \\ &\leq \mathbb{E}(x_0 + Tkl)^p + \frac{p(p-1)}{2} \sigma^2 \int_0^t (\mathbb{E}v_{s \wedge \theta_R}^p)^{\frac{p-1}{p}} ds \end{aligned}$$

Using now a Gronwall type theorem (see [19], Theorem 1, p. 360), we arrive at

$$\mathbb{E}v_{t \wedge \theta_R}^p \leq \left([\mathbb{E}(x_0 + Tkl)^p]^{\frac{p-1}{p}} + \frac{T}{2}(p-1)\sigma^2 \right)^{\frac{p}{p-1}}. \quad (12)$$

But $\mathbb{E}v_{t \wedge \theta_R}^p = \mathbb{E}(v_{t \wedge \theta_R}^p \mathbb{I}_{\{\theta_R \geq t\}}) + R^p P(\theta_R < t)$. That means that $P(t \wedge \theta_R < t) = P(\theta_R < t) \rightarrow 0$ as $R \rightarrow \infty$ so $t \wedge \theta_R \rightarrow t$ in probability and noting that θ_R increases as R increases we have that $t \wedge \theta_R \rightarrow t$ almost surely too, as $R \rightarrow \infty$. Going back to (4) and using Fatou's lemma we obtain,

$$\mathbb{E}v_t^p \leq \left([\mathbb{E}(x_0 + Tkl)^p]^{\frac{p-1}{p}} + \frac{T(p-1)\sigma^2}{2} \right)^{\frac{p}{p-1}}$$

We have assume in our assumptions that $\mathbb{E}x_0^p < \infty$ in order the term $\mathbb{E}(x_0 + Tkl)^p$ to be well posed. The same holds for x_t . \square

Consider now the following auxiliary stochastic process,

$$h_t = x_0 + \int_0^t (kl - k_2 y_s - k_1 y_{\bar{s}}) ds + \sigma \int_0^t \sqrt{y_s} dW_s, \quad t \in (t_k, t_{k+1}].$$

Lemma 5 *We have the following estimates,*

$$\begin{aligned} \mathbb{E}|h_s - y_s|^2 &\leq C_1 \Delta^2 \text{ for any } s \in [0, T] \\ \mathbb{E}|h_s - y_{t_k}|^2 &\leq C_2 \Delta \text{ when } s \in [t_k, t_{k+1}] \\ \mathbb{E}|h_s|^2 &< A, \text{ for any } s \in [0, T]. \end{aligned}$$

Proof. Noting that

$$h_t - y_t = \int_t^{t_{k+1}} (k_1 l - k_1 y_{\bar{s}}) ds$$

we can easily take the results. \square

Theorem 2 *If Assumption A holds then*

$$\mathbb{E}|x_t - y_t|^2 \leq C \frac{1}{\sqrt{\ln n}}$$

for any $t \in [0, T]$.

Proof.

Applying Ito's formula on $|x_t - h_t|^2$ we obtain

$$\mathbb{E}|x_t - h_t|^2 \leq \int_0^t \mathbb{E} (2k_1|h_s - x_s||x_s - y_s| + 2k_2|h_s - x_s||x_s - y_s| + \sigma^2|x_s - y_s|) ds \quad (13)$$

Let us estimate the above quantities. It is easy to see that, using Young inequality,

$$\mathbb{E}|x_s - h_s||y_s - x_s| + \mathbb{E}|x_s - h_s||y_s - x_s| \leq C\mathbb{E}|x_s - h_s|^2 + C\sqrt{\Delta}$$

Summing up we arrive at

$$\mathbb{E}|x_t - h_t|^2 \leq C\sqrt{\Delta} + C \int_0^t \mathbb{E}|x_s - h_s|^2 ds + \sigma^2 \int_0^t \mathbb{E}|x_s - h_s| ds. \quad (14)$$

Therefore, we have to estimate $\mathbb{E}|x_t - h_t|$. Let the non increasing sequence $\{e_m\}_{m \in \mathbb{N}}$ with $e_m = e^{-m(m+1)/2}$ and $e_0 = 1$. We introduce the following sequence of smooth approximations of $|x|$, (method of Yamada and Watanabe, [23])

$$\phi_m(x) = \int_0^{|x|} dy \int_0^y \psi_m(u) du,$$

where the existence of the continuous function $\psi_m(u)$ with $0 \leq \psi_m(u) \leq 2/(mu)$ and support in (e_m, e_{m-1}) is justified by $\int_{e_m}^{e_{m-1}} (du/u) = m$. The following relations hold for $\phi_m \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ with $\phi_m(0) = 0$,

$$\begin{aligned} |x| - e_{m-1} &\leq \phi_m(x) \leq |x|, \quad |\phi'_m(x)| \leq 1, \quad x \in \mathbb{R}, \\ |\phi''_m(x)| &\leq \frac{2}{m|x|}, \quad \text{when } e_m < |x| < e_{m-1} \text{ and } |\phi''_m(x)| = 0 \text{ otherwise.} \end{aligned}$$

Applying Ito's formula on $\phi_m(x_t - h_t)$ we obtain

$$\begin{aligned} \mathbb{E}\phi_m(x_t - h_t) &\leq \int_0^t \mathbb{E}\phi'_m(x_s - h_s)(k_1(y_s - x_s) + k_2(y_s - x_s)) ds \\ &\quad + \int_0^t \frac{\sigma^2}{2} \mathbb{E}\phi''_m(x_s - h_s)|x_s - y_s| ds. \end{aligned}$$

We continue by estimating

$$\mathbb{E}\phi'_m(x_s - h_s)(k_1(y_s - x_s) + k_2(y_s - x_s)) \leq C\mathbb{E}|x_s - h_s| + C\sqrt{\Delta}.$$

Next,

$$\mathbb{E}\phi''_m(x_s - h_s)(\sqrt{y_s} - \sigma\sqrt{x_s})^2 \leq \frac{4\sigma^2}{m} + \frac{4\sigma^2}{m} \mathbb{E} \frac{|h_s - y_s|}{e_m} \leq \frac{4\sigma^2}{m} + \frac{C}{m} \frac{\sqrt{\Delta}}{e_m}$$

Therefore,

$$\mathbb{E}|x_t - h_t| \leq e_{m-1} + \frac{4\sigma^2}{m} + C \frac{\sqrt{\Delta}}{me_m} + k \int_0^t \mathbb{E}|x_s - h_s| ds.$$

Use now Gronwall's inequality and substitute in (3) and then again Gronwall's inequality we arrive at

$$\mathbb{E}|x_t - h_t|^2 \leq C\sqrt{\Delta} + C \frac{\sqrt{\Delta}}{me_m} + e_{m-1}.$$

Choosing $m = \sqrt{\ln n^{\frac{1}{3}}}$ we deduce that

$$\mathbb{E}|x_t - h_t|^2 \leq C \frac{1}{\sqrt{\ln n}}$$

But

$$\mathbb{E}|x_t - y_t|^2 \leq 2\mathbb{E}|x_t - h_t|^2 + 2\mathbb{E}|h_t - y_t|^2 \leq C \frac{1}{\sqrt{\ln n}}$$

□

4.2 The case $2kl > 5\sigma^2$

Here, we choose again k_1, k_2 such that $k = k_1 + k_2$ and $d = \frac{4k_2l}{\sigma^2} = [\frac{4kl}{\sigma^2}]$. Our result is as follows.

Proposition 2 *If*

$$\sigma^2 \leq 2kl \text{ and } \frac{1}{16}(\frac{2kl}{\sigma^2} - 1)^2 > 1$$

the following rate of convergence holds, assuming that $x_0 \in \mathbb{R}_+$,

$$\mathbb{E}|x_t - y_t|^2 \leq C\Delta.$$

Proof.

Define the process

$$\gamma(t) = \int_0^t \frac{ds}{(\sqrt{x_s} + \sqrt{h_s})^2},$$

and then the stopping time defined by

$$\tau_l = \inf\{s \in [0, T] : 2\sigma^2\gamma(s) + 3ks \geq l\}.$$

Using Ito's formula on $|x_\tau - h_\tau|^2$ with τ a stopping time, we obtain

$$\begin{aligned} \mathbb{E}(x_\tau - h_\tau)^2 &\leq \int_0^\tau (2k_1\mathbb{E}|x_s - h_s||y_s - x_s| + 2k_2\mathbb{E}|h_s - x_s||x_s - y_s| + \sigma^2\mathbb{E}|\sqrt{y_s} - \sqrt{x_s}|^2) ds \\ &\leq C\Delta + \int_0^\tau (3k\mathbb{E}|x_s - h_s|^2 + \sigma^2\mathbb{E}|\sqrt{y_s} - \sqrt{x_s}|^2) ds \end{aligned}$$

Now we work on

$$\int_0^\tau \mathbb{E}|\sqrt{y_s} - \sqrt{x_s}|^2 ds \leq \int_0^\tau 2\mathbb{E}|\sqrt{x_s} - \sqrt{h_s}|^2 + 2\mathbb{E}|\sqrt{h_s} - \sqrt{y_s}|^2 ds \leq 2 \int_0^\tau \mathbb{E}|\sqrt{x_s} - \sqrt{h_s}|^2 ds + C\Delta.$$

But

$$\int_0^\tau \mathbb{E}|\sqrt{x_s} - \sqrt{h_s}|^2 ds = \int_0^\tau \mathbb{E} \frac{|x_s - h_s|^2}{(\sqrt{x_s} + \sqrt{h_s})^2} ds.$$

Therefore,

$$\mathbb{E}(x_\tau - h_\tau)^2 \leq C\Delta + \mathbb{E} \int_0^\tau |x_s - h_s|^2 (3ks + 2\sigma^2\gamma_s)' ds. \quad (15)$$

Now, for $\tau = \tau_l$, we use the change of variables setting $u = 3ks + 2\sigma^2\gamma_s$ and therefore $s = \tau_u$ obtaining,

$$\mathbb{E}(x_{\tau_l} - h_{\tau_l})^2 \leq C\Delta + \int_0^l \mathbb{E}|x_{\tau_u} - h_{\tau_u}|^2 du.$$

Using Gronwall's inequality we obtain,

$$\mathbb{E}|x_{\tau_l} - h_{\tau_l}|^2 \leq Ce^l \Delta. \quad (16)$$

Going back to (4), for $\tau = t \in [0, T]$, we have under the change of variables $u = 2\sigma^2\gamma_s + 3ks$,

$$\begin{aligned} \mathbb{E}(x_t - h_t)^2 &\leq C\Delta + \mathbb{E} \int_0^{3kT + 2\sigma^2\gamma_T} |x_{\tau_u} - h_{\tau_u}|^2 du \\ &\leq C\Delta + \int_0^\infty \mathbb{E}(\mathbb{I}_{\{3kT + 2\sigma^2\gamma_T \geq u\}} |x_{\tau_u} - h_{\tau_u}|^2) du. \end{aligned} \quad (17)$$

Noting that

$$\begin{aligned}
& \int_0^\infty \mathbb{E}(\mathbb{I}_{\{3kT+2\sigma^2\gamma_T \geq u\}} |x_{\tau_u} - h_{\tau_u}|^2) du \\
& \leq \int_0^{3kT} \mathbb{E}|x_{\tau_u} - h_{\tau_u}|^2 du + \int_{3kT}^\infty \mathbb{P}(3kT + 2\sigma^2\gamma_T \geq u) \mathbb{E}(|x_{\tau_u} - h_{\tau_u}|^2 \mid \{3kT + 2\sigma^2\gamma_T \geq u\}) du, \\
& \leq C\Delta + \int_0^\infty \mathbb{P}(2\sigma^2\gamma_T \geq u) \mathbb{E}|x_{\tau_u} - h_{\tau_u}|^2 du
\end{aligned}$$

and then we arrive using (5)

$$\mathbb{E}(x_t - h_t)^2 \leq C\Delta \left(1 + C \int_0^\infty \mathbb{P}(2\sigma^2\gamma_T \geq u) e^u du\right).$$

We will estimate now the following,

$$\mathbb{P}(2\sigma^2\gamma_T \geq u) \leq \frac{1}{e^{mu}} \mathbb{E}(e^{2m\sigma^2\gamma_T})$$

Choose $m = \frac{1}{16}(\frac{2kl}{\sigma^2} - 1)^2$ and use Thm. 3.1 of [17] to end the proof. \square

5 An explicit scheme for the two factor CIR model based on exact simulation

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a complete probability space with a filtration and let two independent Wiener processes $(W_t^{1,2})_{t \geq 0}$ defined on this space. Here we consider the following two factor CIR model (see [20], p. 420),

$$\begin{aligned}
x_1(t) &= x_1(0) + \int_0^t (k - \lambda_{11}x_1(s) + \lambda_{12}x_2(s))ds + \int_0^t \sigma_1 \sqrt{x_1(s)} dW_s^1, \\
x_2(t) &= x_2(0) + \int_0^t (l - \lambda_{21}x_2(s) + \lambda_{22}x_1(s))ds + \int_0^t \sigma_2 \sqrt{x_2(s)} dW_s^2
\end{aligned}$$

This kind of model is widely used in financial mathematics. If one wants to calculate complicate expressions of the solution of the above system maybe the only way is to approximate it numerically. In this case, the numerical scheme should be positivity preserving and the usual Euler scheme does not have this property. For more details about the use of this model in financial mathematics one can see for example [20].

In the following two sections we will propose two different, explicit and positivity preserving numerical schemes.

Our starting point is the exact simulation for the CIR process for some specific parameters. Consider the CIR process, and let $0 = t_0 < t_1 < \dots < t_n = T$, setting $\Delta = \frac{T}{n}$,

$$x_t = x_0 + \int_0^t (kl - kx_s)ds + \sigma \int_0^t \sqrt{x_s} dW_s.$$

If $d = \frac{4kl}{\sigma^2} \in \mathbb{N}$ then we can simulate this process exactly (see [10]), p. 133). Indeed, the exact simulation is given by

$$r(t_{i+1}) = \sum_{j=1}^d \left(e^{-\frac{1}{2}k\Delta} \sqrt{\frac{r(t_i)}{d}} + \frac{\sigma}{2} \sqrt{\frac{1}{k}(-e^{-k\Delta})} Z_{i+1}^{(j)} \right)^2,$$

where $(Z_i^{(1)}, \dots, Z_i^{(d)})$ are standard normal d -vectors, independent for different values of i . Therefore, the idea (see [11]) is to split a part of the drift term and the remaining drift coefficient will be

such that we can simulate it exactly. Then, we will study the error produced by this splitting. For the two factor CIR model there is one more difficulty. In each equation there exists an unknown stochastic process which appears on the other. In this situation we will use the main idea of [12] and discretize every part of the first stochastic differential equation that contains the unknown stochastic process which contained in the second equation and vice versa. In this way we arrive to two stochastic differential equations that contains only one unknown stochastic process. For another positivity preserving numerical scheme for one factor CIR model see [1].

We propose the following decomposition,

$$\begin{aligned} y_1(t) &= y_1(t_k) + \Delta\lambda_{12}y_2(t_k) + \Delta k_1 + \int_{t_k}^t (k_2 - \lambda_{11}y_1(s))ds \\ &+ \sigma_1 \int_{t_k}^t \sqrt{y_1(s)}dW_s^1, \quad t \in (t_k, t_{k+1}] \\ y_2(t) &= y_2(t_k) + \Delta\lambda_{22}y_1(t_k) + \Delta l_1 + \int_{t_k}^t (l_2 - \lambda_{21}y_2(s))ds \\ &+ \sigma_2 \int_{t_k}^t \sqrt{y_2(s)}dW_s^2, \quad t \in (t_k, t_{k+1}] \end{aligned}$$

where $\frac{4k_2}{\sigma_1^2} = [\frac{4k}{\sigma_1^2}]$, $\frac{4l_2}{\sigma_2^2} = [\frac{4l}{\sigma_2^2}]$ and by $[\cdot]$ we denote the integer part. We see that the above sdes are not really a system and in each equation only one unknown stochastic process appears. Therefore, in each step, we can simulate exactly the stochastic process y_1, y_2 .

Let us write in a more compact form our numerical scheme, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} y_1(t) &= x_1(0) + \int_0^t (k - \lambda_{11}y_1(s) + \lambda_{12}y_2(\hat{s})) ds + (t_{k+1} - t)(k_1 + \lambda_{12}y_2(t_k)) \\ &+ \sigma_1 \int_0^t \sqrt{y_1(s)}dW_s^1, \\ y_2(t) &= x_2(0) + \int_0^t (l - \lambda_{21}y_2(s) + \lambda_{22}y_1(\hat{s})) ds + (t_{k+1} - t)(l_1 + \lambda_{22}y_1(t_k)) \\ &+ \sigma_2 \int_0^t \sqrt{y_2(s)}dW_s^2, \end{aligned}$$

where $\hat{s} = t_k$ when $s \in [t_k, t_{k+1}]$. Our first result is to obtain the moment bounds for the true and the approximate solution.

Assumption A Assume that $x_1(0), x_2(0) \in \mathbb{R}_+$ and that $d_1 = \frac{4k}{\sigma_1^2} > 1$, $d_2 = \frac{4l}{\sigma_2^2} > 1$.

Below we will give the moment bounds for the true and the approximate solution. However, for the approximate solution it seems that we need to bound it uniformly as we did, for example in [13].

Lemma 6 *Under Assumption A we have*

$$\mathbb{E}(\sup_{0 \leq t \leq T} (y_1(t)^2 + y_2(t)^2)) < C, \quad \mathbb{E}x_1(t)^2 + x_2(t)^2 < C.$$

Proof. We easily see that

$$\begin{aligned} 0 \leq y_1(t) \leq v_1(t) &= x_1(0) + Tk + \Delta\lambda_{12}x_2(0) + \int_0^t \lambda_{12}(y_2(\hat{s}) + y_2(t_k))ds \\ &+ \sigma_1 \int_0^t \sqrt{y_1(s)}dW_s^1, \end{aligned}$$

$$\begin{aligned}
0 \leq y_2(t) \leq v_2(t) = x_2(0) &+ Tl + \Delta \lambda_{22} x_1(0) + \int_0^t \lambda_{22} (y_1(\hat{s}) + y_1(t_k)) ds \\
&+ \sigma_2 \int_0^t \sqrt{y_2(s)} dW_s^2,
\end{aligned}$$

We have used that $(t_{k+1} - t)\lambda_{12}y_2(t_k) \leq \Delta\lambda_{12}x_2(0)$ when $t_k = t_0$ and $(t_{k+1} - t)\lambda_{12}y_2(t_k) \leq \int_0^t \lambda_{12}y_2(t_k)ds$ when $t_k = t_1, t_2, \dots$ and therefore $t > \Delta$. Thus,

$$(t_{k+1} - t)\lambda_{12}y_2(t_k) \leq \Delta\lambda_{12}x_2(0) + \int_0^t \lambda_{12}y_2(t_k)ds,$$

for any $t \in [0, T]$. The same holds for the $y_1(t_k)$.

Consider the stopping time $\tau = \inf\{t \in [0, T] : y_1(t) > R \text{ or } y_2(t) > R\}$. Then, we can write,

$$\begin{aligned}
v_1^2(t \wedge \tau) &\leq C + C \int_0^{t \wedge \tau} v_2^2(\hat{s} \wedge \tau) + v_2^2(t_k \wedge \tau) ds + C \left| \int_0^{t \wedge \tau} \sqrt{y_1(s \wedge \tau)} dW_s^1 \right|^2, \\
v_2^2(t \wedge \tau) &\leq C + C \int_0^{t \wedge \tau} v_1^2(\hat{s} \wedge \tau) + v_1^2(t_k \wedge \tau) ds + C \left| \int_0^{t \wedge \tau} \sqrt{y_2(s \wedge \tau)} dW_s^2 \right|^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
\sup_{0 \leq t \leq r} (v_1^2(t \wedge \tau) + v_2^2(t \wedge \tau)) &\leq C + C \int_0^r (v_1^2(\hat{s} \wedge \tau) + v_2^2(\hat{s} \wedge \tau) + v_2^2(t_k \wedge \tau) + v_1^2(t_k \wedge \tau)) ds \\
&+ C \left(\sup_{0 \leq t \leq r} \left| \int_0^{t \wedge \tau} \sqrt{y_1(s)} dW_s^1 \right|^2 + \sup_{0 \leq t \leq r} \left| \int_0^{t \wedge \tau} \sqrt{y_2(s)} dW_s^2 \right|^2 \right).
\end{aligned}$$

Taking expectations and using Doob's martingale inequality we arrive at

$$\begin{aligned}
\mathbb{E} \left(\sup_{0 \leq t \leq r} (v_1^2(t) + v_2^2(t)) \right) &\leq C + C \int_0^r \mathbb{E}((v_1^2(\hat{s} \wedge \tau) + v_2^2(\hat{s} \wedge \tau) + v_2^2(t_k \wedge \tau) + v_1^2(t_k \wedge \tau)) ds \\
&+ C \int_0^r \mathbb{E}(v_1(s \wedge \tau) + v_2(s \wedge \tau)) ds \\
&\leq C + C \int_0^r \left(\mathbb{E} \sup_{0 \leq \beta \leq s} (v_1^2(\beta \wedge \tau) + v_2^2(\beta \wedge \tau)) + \sqrt{\mathbb{E} \left(\sup_{0 \leq \beta \leq s} (v_1^2(\beta \wedge \tau) + v_2^2(\beta \wedge \tau)) \right)} \right) ds.
\end{aligned}$$

Setting now $u(r) = \mathbb{E}(\sup_{0 \leq t \leq r} (v_1^2(t \wedge \tau) + v_2^2(t \wedge \tau)))$ and using a generalized Gronwall inequality (see [19], Theorem 1, p. 360) we deduce that

$$u(r) \leq C, \quad r \in [0, T]$$

with C independent of R . Taking the limit as $R \rightarrow \infty$ and using Fatou's lemma we take our result. The same holds for x_1, x_2 . □

We will use later the auxiliary stochastic processes,

$$\begin{aligned}
h_1(t) &= x_1(0) + \int_0^t (k - \lambda_{11}y_1(s) + \lambda_{12}y_2(\hat{s})) ds + \sigma_1 \int_0^t \sqrt{y_1(s)} dW_s^1, \\
h_2(t) &= x_2(0) + \int_0^t (l - \lambda_{21}y_2(s) + \lambda_{22}y_1(\hat{s})) ds + \sigma_2 \int_0^t \sqrt{y_2(s)} dW_s^2
\end{aligned}$$

We shall show below that $h_{1,2}(t)$ and $y_{1,2}(t)$ remain close.

Lemma 7 *Under Assumption A we have, for all $t \in [0, T]$,*

$$\begin{aligned} \mathbb{E}|h_{1,2}(t) - y_{1,2}(t)|^2 &\leq C\Delta^2 \\ \mathbb{E}|h_{1,2}(t) - y_{1,2}(t_k)|^2 &\leq C\Delta \text{ when } t \in [t_k, t_{k+1}] \\ \mathbb{E}h_{1,2}^2(t) &\leq C. \end{aligned}$$

Proof. It is easy to see that

$$\mathbb{E}|y_{1,2}(t) - y_{1,2}(\hat{t})|^2 \leq C\Delta.$$

Moreover, noting that

$$\mathbb{E}|y_{1,2}(t) - h_{1,2}(t)| \leq C\Delta^2,$$

we obtain the other results. □

5.1 The general case $d_1 \geq 1, d_2 \geq 1$

In this section we assume that $d_1 > 1$ and $d_2 > 1$ and we will prove that the rate of convergence is at least logarithmic. If $d_1 = 1$ for example we can simulate x_1 exactly therefore we work on the case where $d_1 > 1$ and $d_2 > 1$.

Theorem 3 *If Assumption A holds then*

$$\mathbb{E}(|x_1(t) - y_1(t)|^2 + |x_2(t) - y_2(t)|^2) \leq C \frac{1}{\sqrt{\ln n}}$$

for any $t \in [0, T]$.

Proof.

Applying Ito's formula on $|x_1(t) - h_1(t)|^2$ we obtain

$$\begin{aligned} &\mathbb{E}|x_1(t) - h_1(t)|^2 \\ &\leq \int_0^t \mathbb{E}(2\lambda_{11}|x_1(s) - h_1(s)||y_1(s) - x_1(s)| + 2\lambda_{12}|x_1(s) - h_1(s)||x_2(s) - y_2(\hat{s})| \\ &\quad + \sigma_1^2|x_1(s) - y_1(s)|)ds \end{aligned} \tag{18}$$

Using Young inequality, we deduce

$$\begin{aligned} &\mathbb{E}|x_1(s) - h_1(s)||y_1(s) - x_1(s)| + \mathbb{E}|x_1(s) - h_1(s)||y_2(\hat{s}) - x_2(s)| \\ &\leq C(\mathbb{E}|x_1(s) - h_1(s)|^2 + \mathbb{E}|x_2(s) - h_2(s)|^2) + C\Delta \end{aligned}$$

Summing up we arrive at

$$\begin{aligned} &\mathbb{E}|x_1(t) - h_1(t)|^2 \\ &\leq C\sqrt{\Delta} + C \int_0^t \mathbb{E}(|x_1(s) - h_1(s)|^2 + |x_2(s) - h_2(s)|^2)ds \\ &\quad + \sigma_1^2 \int_0^t \mathbb{E}|x_1(s) - h_1(s)|ds. \end{aligned} \tag{19}$$

Setting $v^2(t) = |x_1(s) - h_1(s)|^2 + |x_2(s) - h_2(s)|^2$, using Ito's formula as before on $|x_2(t) - h_2(t)|^2$ and adding the results we arrive at

$$\mathbb{E}v^2(t) \leq C\sqrt{\Delta} + C \int_0^t \mathbb{E}v^2(s)ds + (\sigma_1^2 + \sigma_2^2) \int_0^t \mathbb{E}(|x_1(s) - h_1(s)| + |x_2(s) - h_2(s)|)ds.$$

Therefore, we have to estimate $\mathbb{E}|x_1(t) - h_1(t)|$ and $\mathbb{E}|x_2(t) - h_2(t)|$. Let the non increasing sequence $\{e_m\}_{m \in \mathbb{N}}$ with $e_m = e^{-m(m+1)/2}$ and $e_0 = 1$. We introduce the following sequence of smooth approximations of $|x|$, (method of Yamada and Watanabe, [23])

$$\phi_m(x) = \int_0^{|x|} dy \int_0^y \psi_m(u) du,$$

where the existence of the continuous function $\psi_m(u)$ with $0 \leq \psi_m(u) \leq 2/(mu)$ and support in (e_m, e_{m-1}) is justified by $\int_{e_m}^{e_{m-1}} (du/u) = m$. The following relations hold for $\phi_m \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ with $\phi_m(0) = 0$,

$$\begin{aligned} |x| - e_{m-1} &\leq \phi_m(x) \leq |x|, \quad |\phi'_m(x)| \leq 1, \quad x \in \mathbb{R}, \\ |\phi''_m(x)| &\leq \frac{2}{m|x|}, \quad \text{when } e_m < |x| < e_{m-1} \text{ and } |\phi''_m(x)| = 0 \text{ otherwise.} \end{aligned}$$

Applying Ito's formula on $\phi_m(x_1(t) - h_1(t))$ we obtain

$$\begin{aligned} \mathbb{E}\phi_m(x_1(t) - h_1(t)) &\leq \int_0^t \mathbb{E}|\phi'_m(x_1(s) - h_1(s))|(\lambda_{11}|y_1(s) - x_1(s)| + \lambda_{12}|x_2(s) - y_2(\hat{s})|)ds \\ &\quad + \int_0^t \frac{\sigma_1^2}{2} \mathbb{E}|\phi''_m(x_1(s) - h_1(s))||x_1(s) - y_1(s)|ds. \end{aligned}$$

We continue by estimating,

$$\begin{aligned} &\mathbb{E}|\phi'_m(x_1(s) - h_1(s))|(\lambda_{11}|y_1(s) - x_1(s)| + \lambda_{12}|x_2(s) - y_2(\hat{s})|) \\ &\leq C\mathbb{E}(|x_1(s) - h_1(s)| + |x_2(s) - h_2(s)|) + C\sqrt{\Delta}, \end{aligned}$$

and

$$\mathbb{E}|\phi''_m(x_1(s) - h_1(s))||x_1(s) - y_1(s)| \leq \frac{2}{m} + \frac{2}{me_m} \mathbb{E}|h_1 - y_1| \leq \frac{2}{m} + \frac{2C}{me_m} \sqrt{\Delta}$$

Therefore,

$$\mathbb{E}|x_1(t) - h_1(t)| \leq e_{m-1} + \frac{4\sigma_1^2}{m} + C\frac{\sqrt{\Delta}}{me_m} + C \int_0^t \mathbb{E}(|x_1(s) - h_1(s)| + |x_2(s) - h_2(s)|)ds.$$

Now, we do exact the same for $|x_2(s) - h_2(s)|$, adding the results and setting $u(t) = |x_1(s) - h_1(s)| + |x_2(s) - h_2(s)|$ we arrive at

$$\mathbb{E}u(t) \leq 2e_{m-1} + \frac{4(\sigma_1^2 + \sigma_2^2)}{m} + C\frac{\sqrt{\Delta}}{me_m} + C \int_0^t \mathbb{E}u(s)ds$$

Use now Gronwall's inequality and substitute in (19) and then again Gronwall's inequality we arrive at

$$\mathbb{E}v^2(t) \leq C\sqrt{\Delta} + C\frac{\sqrt{\Delta}}{me_m} + e_{m-1}.$$

Choosing $m = \sqrt{\ln n^{\frac{1}{3}}}$ we deduce that

$$\mathbb{E}v^2(t) \leq C\frac{1}{\sqrt{\ln n}}$$

But

$$\begin{aligned} \mathbb{E}(|x_1(t) - y_1(t)|^2 + |x_2(t) - y_2(t)|^2) &\leq 2\mathbb{E}v^2(t) + 2\mathbb{E}(|h_1(t) - y_1(t)|^2 + |h_2(t) - y_2(t)|^2) \\ &\leq C\frac{1}{\sqrt{\ln n}} \end{aligned}$$

□

5.2 Polynomial order of convergence

In this section we will prove that the order of convergence is at least 1/2 under further conditions on parameters and to do this we state first a proposition in which we show that the true solutions x_1, x_2 has exponential inverse moment bounds.

Consider the following CIR processes.

$$\begin{aligned} z_1(t) &= x_1(0) + \int_0^t (k - \lambda_{11} z_1(s)) ds + \sigma_1 \int_0^t \sqrt{z_1(s)} dW_s^1, \\ z_2(t) &= x_2(0) + \int_0^t (l - \lambda_{21} z_2(s)) ds + \sigma_2 \int_0^t \sqrt{z_2(s)} dW_s^2. \end{aligned}$$

Assumption B Assume that there exists some strictly positive constants $L_1(\sigma_1, k), L_2(\sigma_2, l)$ such that

$$\mathbb{E} \exp \left(\int_0^T \frac{L_1}{z_1(s)} ds \right) < \infty, \quad \mathbb{E} \exp \left(\int_0^T \frac{L_2}{z_2(s)} ds \right) < \infty.$$

One can see [15], [17], [6] for a discussion on this assumption.

Proposition 3 Suppose that Assumption A and B hold. Then, the following bounds are true,

$$\mathbb{E} \exp \left(\int_0^T \frac{L_1}{x_1(s)} ds \right) < \infty, \quad \mathbb{E} \exp \left(\int_0^T \frac{L_2}{x_2(s)} ds \right) < \infty.$$

Proof. From the comparison theorem (see [16], prop. 5.2.18) we know that $x_1(t) \geq z_1(t)$ with

$$z_1(t) = x_1(0) + \int_0^t (k - \lambda_{11} z_1(s)) ds + \sigma_1 \int_0^t \sqrt{z_1(s)} dW_s^1.$$

Therefore, since for z_1 we have exponential inverse moment bounds we take the result. The same holds for x_2 . \square

Proposition 4 Assume assumptions A and B. If $\frac{L_1}{4(\sigma_1^2 + \sigma_2^2)} \geq 1$ and $\frac{L_2}{4(\sigma_1^2 + \sigma_2^2)} \geq 1$ the following rate of convergence holds, assuming that $x_0 \in \mathbb{R}_+$,

$$\mathbb{E} |x_1(t) - y_1(t)|^2 + |x_2(t) - y_2(t)|^2 \leq C \Delta.$$

Proof.

Define the processes,

$$\begin{aligned} \gamma_1(t) &= \int_0^t \frac{ds}{(\sqrt{x_1(s)} + \sqrt{h_1(s)})^2} \\ \gamma_2(t) &= \int_0^t \frac{ds}{(\sqrt{x_2(s)} + \sqrt{h_2(s)})^2} \\ \gamma(t) &= \gamma_1(t) + \gamma_2(t) \end{aligned}$$

and then the stopping times defined by

$$\begin{aligned} \tau_l^1 &= \inf \{ s \in [0, T] : 4(\sigma_1^2 + \sigma_2^2) \gamma_1(s) + \frac{K}{2} s \geq l \}, \\ \tau_l^2 &= \inf \{ s \in [0, T] : 4(\sigma_1^2 + \sigma_2^2) \gamma_2(s) + \frac{K}{2} s \geq l \}, \\ \tau_l &= \inf \{ s \in [0, T] : 4(\sigma_1^2 + \sigma_2^2) \gamma(s) + K s \geq l \} \end{aligned}$$

for some fixed $K > 0$.

Using Ito's formula on $|x_1(\tau) - h_1(\tau)|^2$ with τ a stopping time, we obtain

$$\begin{aligned}
\mathbb{E}(x_1(\tau) - h_1(\tau))^2 &\leq \int_0^\tau (\mathbb{E}2\lambda_{11}|x_1(s) - h_1(s)||y_1(s) - x_1(s)| \\
&\quad + 2\lambda_{12}|x_1(s) - h_1(s)||x_2(s) - y_2(\hat{s})| + \sigma_1^2\mathbb{E}|\sqrt{y_1(s)} - \sqrt{x_1(s)}|^2)ds \\
&\leq \int_0^\tau 3\lambda_{11}\mathbb{E}|x_1(s) - h_1(s)|^2 + 2\lambda_{11}\mathbb{E}|h_1(s) - y_1(s)|^2 \\
&\quad + \lambda_{12}\mathbb{E}|x_1(s) - h_1(s)|^2 + 2\lambda_{12}\mathbb{E}|x_2(s) - h_2(s)|^2 \\
&\quad + 2\lambda_{12}\mathbb{E}|h_2(s) - y_2(\hat{s})|^2 + \sigma_1^2\mathbb{E}|\sqrt{y_1(s)} - \sqrt{x_1(s)}|^2)ds \\
&\leq C\Delta + \int_0^\tau (3\lambda_{11} + \lambda_{12})\mathbb{E}(|x_1(s) - h_1(s)|^2 + |x_2(s) - h_2(s)|^2) \\
&\quad + \sigma_1^2\mathbb{E}|\sqrt{y_1(s)} - \sqrt{x_1(s)}|^2)ds
\end{aligned}$$

The last term can be expressed as

$$\begin{aligned}
\int_0^\tau \mathbb{E}|\sqrt{y_1(s)} - \sqrt{x_1(s)}|^2 ds &\leq \int_0^\tau 2\mathbb{E}|\sqrt{x_1(s)} - \sqrt{h_1(s)}|^2 + 2\mathbb{E}|\sqrt{h_1(s)} - \sqrt{y_1(s)}|^2 ds \\
&\leq 2 \int_0^\tau \mathbb{E}|\sqrt{x_1(s)} - \sqrt{h_1(s)}|^2 ds + C\Delta.
\end{aligned}$$

But

$$\int_0^\tau \mathbb{E}|\sqrt{x_1(s)} - \sqrt{h_1(s)}|^2 ds = \int_0^\tau \mathbb{E} \frac{|x_1(s) - h_1(s)|^2}{(\sqrt{x_1(s)} + \sqrt{h_1(s)})^2} ds.$$

Doing exactly the same work on $|x_2(\tau) - h_2(\tau)|^2$, adding the results and setting $v^2(\tau) = |x_1(\tau) - h_1(\tau)|^2 + |x_2(\tau) - h_2(\tau)|^2$ we get,

$$\begin{aligned}
\mathbb{E}v^2(\tau) &\leq C\Delta + \int_0^\tau \mathbb{E}(Kv^2(s) + \frac{2\sigma_1^2|x_1(s) - h_1(s)|^2}{(\sqrt{x_1(s)} + \sqrt{h_1(s)})^2} + \frac{2\sigma_2^2|x_2(s) - h_2(s)|^2}{(\sqrt{x_2(s)} + \sqrt{h_2(s)})^2})ds \\
&\leq C\Delta + \int_0^\tau \mathbb{E}(Ks + 4(\sigma_1^2 + \sigma_2^2)\gamma_s)'v_s^2 ds
\end{aligned} \tag{20}$$

Now, for $\tau = \tau_l$, we use the change of variables setting $u = 4(\sigma_1^2 + \sigma_2^2)\gamma(s) + Ks$ and therefore $s = \tau_u$ obtaining,

$$\mathbb{E}v_{\tau_l}^2 \leq C\Delta + \int_0^l \mathbb{E}v_{\tau_u}^2 du.$$

Using Gronwall's inequality we obtain,

$$\mathbb{E}v_{\tau_l}^2 \leq Ce^l \Delta. \tag{21}$$

Now we rewrite (20) as follows,

$$\begin{aligned}
\mathbb{E}v^2(\tau) &\leq C\Delta + \int_0^\tau \mathbb{E}(\frac{K}{2}s + 4(\sigma_1^2 + \sigma_2^2)\gamma_1(s))'v^2(s)ds \\
&\quad + \int_0^\tau \mathbb{E}(\frac{K}{2}s + 4(\sigma_1^2 + \sigma_2^2)\gamma_2(s))'v^2(s)ds
\end{aligned} \tag{22}$$

For $\tau = t \wedge \tau_l \in [0, T]$ in (22), we have under the change of variables $u = 4(\sigma_1^2 + \sigma_2^2)\gamma_1(s) + \frac{K}{2}s$, for

the first integral, and the change of variables $u = 4(\sigma_1^2 + \sigma_2^2)\gamma_2(s) + \frac{K}{2}s$ for the second integral,

$$\begin{aligned}
\mathbb{E}v^2(t \wedge \tau_l) &\leq C\Delta + \mathbb{E} \int_0^{\frac{K}{2}T + 4(\sigma_1^2 + \sigma_2^2)\gamma_1(T)} v^2(\tau_u^1 \wedge \tau_u) du \\
&\quad + \mathbb{E} \int_0^{\frac{K}{2}T + 4(\sigma_1^2 + \sigma_2^2)\gamma_2(T)} v^2(\tau_u^2 \wedge \tau_u) du \\
&\leq C\Delta + \int_0^\infty \mathbb{E} \left(\mathbb{I}_{\{\frac{K}{2}T + 4(\sigma_1^2 + \sigma_2^2)\gamma_1(T) \geq u\}} v^2(\tau_u^1 \wedge \tau_u) \right) du \\
&\quad + \int_0^\infty \mathbb{E} \left(\mathbb{I}_{\{\frac{K}{2}T + 4(\sigma_1^2 + \sigma_2^2)\gamma_2(T) \geq u\}} v^2(\tau_u^2 \wedge \tau_u) \right) du. \tag{23}
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_0^\infty \mathbb{E} \left(\mathbb{I}_{\{\frac{K}{2}T + 4(\sigma_1^2 + \sigma_2^2)\gamma_1(T) \geq u\}} v^2(\tau_u^1 \wedge \tau_u) \right) du \\
&\leq \int_0^{\frac{K}{2}T} \mathbb{E}v^2(\tau_u^1 \wedge \tau_u) du \\
&\quad + \int_{\frac{K}{2}T}^\infty \mathbb{P}\left(\frac{K}{2}T + 4(\sigma_1^2 + \sigma_2^2)\gamma_1(T) \geq u\right) \mathbb{E} \left(v^2(\tau_u^1 \wedge \tau_u) \mid \left\{ \frac{K}{2}T + 4(\sigma_1^2 + \sigma_2^2)\gamma_1(T) \geq u \right\} \right) du, \\
&\leq C\Delta + \int_0^\infty \mathbb{P}(4(\sigma_1^2 + \sigma_2^2)\gamma_1(T) \geq u) \mathbb{E}v^2(\tau_u^1 \wedge \tau_u) du
\end{aligned}$$

and then, with exactly the same arguments for the integral involving $\gamma_2(t)$, we arrive using (21)

$$\mathbb{E}v^2(t \wedge \tau_l) \leq C\Delta \left(1 + C \int_0^\infty \mathbb{P}(4(\sigma_1^2 + \sigma_2^2)\gamma_1(T) \geq u) e^u du + \int_0^\infty \mathbb{P}(4(\sigma_1^2 + \sigma_2^2)\gamma_2(T) \geq u) e^u du \right).$$

The probability,

$$\mathbb{P}(4(\sigma_1^2 + \sigma_2^2)\gamma_1(T) \geq u) \leq \frac{1}{e^{mu}} \mathbb{E}(e^{4m(\sigma_1^2 + \sigma_2^2)\gamma_1(T)}),$$

and the same holds for the probability involving $\gamma_2(t)$. Choose $m_i = \frac{L_i}{4(\sigma_1^2 + \sigma_2^2)}$, for $i = 1, 2$ and use Proposition 3 to deduce that

$$\mathbb{E}v^2(t \wedge \tau_l) \leq C\Delta.$$

Using Fatou's lemma for $l \rightarrow \infty$ we take the result. \square

6 A second explicit numerical scheme

We will propose a different numerical scheme below,

$$\begin{aligned}
y_1(t_{k+1}) &= \left(\frac{\sigma_1}{2}(W_{t_{k+1}}^1 - W_{t_k}^1) + \sqrt{y_1(t_k)(1 - \lambda_{11}\Delta) + \Delta\lambda_{12}y_2(t_k) + \Delta(k - \frac{\sigma_1^2}{4})} \right)^2, \\
y_2(t_{k+1}) &= \left(\frac{\sigma_2}{2}(W_{t_{k+1}}^2 - W_{t_k}^2) + \sqrt{y_2(t_k)(1 - \lambda_{21}\Delta) + \Delta\lambda_{22}y_1(t_k) + \Delta(l - \frac{\sigma_2^2}{4})} \right)^2.
\end{aligned}$$

Knowing $y_1(t_0) = x_1(0)$, $y_2(t_0) = x_2(0)$ we obtain explicitly and parallel the $y_1(t_1)$, $y_2(t_2)$ and so on.

We work with the following stochastic processes,

$$\begin{aligned}
y_1(t) &= \left(\frac{\sigma_1}{2}(W_t^1 - W_{t_k}^1) + \sqrt{y_1(t_k)(1 - \lambda_{11}\Delta) + \Delta\lambda_{12}y_2(t_k) + \Delta(k - \frac{\sigma_1^2}{4})} \right)^2 = (z_1(t))^2, \\
y_2(t) &= \left(\frac{\sigma_2}{2}(W_t^2 - W_{t_k}^2) + \sqrt{y_2(t_k)(1 - \lambda_{21}\Delta) + \Delta\lambda_{22}y_1(t_k) + \Delta(l - \frac{\sigma_2^2}{4})} \right)^2 = (z_2(t))^2,
\end{aligned}$$

and in fact with the stochastic differentials obtained by the use of Ito's formula, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} y_1(t) &= y_1(t_k)(1 - \lambda_{11}\Delta) + \Delta\lambda_{12}y_2(t_k) + \Delta(k - \frac{\sigma_1^2}{4}) + \int_{t_k}^t \frac{\sigma_1^2}{4}ds \\ &\quad + \sigma_1 \int_{t_k}^t \text{sgn}(z_1(s))\sqrt{y_1(s)}dW_s^1, \\ y_2(t) &= y_2(t_k)(1 - \lambda_{21}\Delta) + \Delta\lambda_{22}y_1(t_k) + \Delta(l - \frac{\sigma_2^2}{4}) + \int_{t_k}^t \frac{\sigma_2^2}{4}ds \\ &\quad + \sigma_2 \int_{t_k}^t \text{sgn}(z_2(s))\sqrt{y_2(s)}dW_s^2. \end{aligned}$$

The compact forms are, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} y_1(t) &= x_1(0) + \int_0^t (k - \lambda_{11}y_1(\hat{s}) + \lambda_{12}y_2(\hat{s}))ds + \int_t^{t_{k+1}} (k - \frac{\sigma_1^2}{4} - \lambda_{11}y_1(t_k) + \lambda_{12}y_2(t_k))ds \\ &\quad + \sigma_1 \int_0^t \text{sgn}(z_1(s))\sqrt{y_1(s)}dW_s^1, \\ y_2(t) &= x_2(0) + \int_0^t (l - \lambda_{21}y_2(\hat{s}) + \lambda_{22}y_1(\hat{s}))ds + \int_t^{t_{k+1}} (l - \frac{\sigma_2^2}{4} - \lambda_{21}y_2(t_k) + \lambda_{22}y_1(t_k))ds \\ &\quad + \sigma_2 \int_0^t \text{sgn}(z_2(s))\sqrt{y_2(s)}dW_s^2. \end{aligned}$$

Finally, we will use the following auxiliary processes,

$$\begin{aligned} h_1(t) &= x_1(0) + \int_0^t (k - \lambda_{11}y_1(\hat{s}) + \lambda_{12}y_2(\hat{s}))ds + \sigma_1 \int_0^t \text{sgn}(z_1(s))\sqrt{y_1(s)}dW_s^1, \\ h_2(t) &= x_2(0) + \int_0^t (l - \lambda_{21}y_2(\hat{s}) + \lambda_{22}y_1(\hat{s}))ds + \sigma_2 \int_0^t \text{sgn}(z_2(s))\sqrt{y_2(s)}dW_s^2. \end{aligned}$$

Assumption C Assume that $d_1 \geq 1$, $d_2 \geq 1$, $\Delta \leq \frac{1}{\max\{\lambda_{11}, \lambda_{21}\}}$ and $x_0 \in \mathbb{R}_+$.

Lemma 8 Under Assumption C we have

$$\mathbb{E}(\sup_{0 \leq t \leq T} (y_1(t)^2 + y_2(t)^2)) < C$$

Proof. Here, again, we easily see that

$$\begin{aligned} 0 \leq y_1(t) \leq v_1(t) &= x_1(0) + Tk + \Delta\lambda_{12}x_2(0) + \int_0^t \lambda_{12}(y_2(\hat{s}) + y_2(t_k))ds \\ &\quad + \sigma_1 \int_0^t \sqrt{y_1(s)}dW_s^1, \\ 0 \leq y_2(t) \leq v_2(t) &= x_2(0) + Tl + \Delta\lambda_{22}x_1(0) + \int_0^t \lambda_{22}(y_1(\hat{s}) + y_1(t_k))ds \\ &\quad + \sigma_2 \int_0^t \sqrt{y_2(s)}dW_s^2, \end{aligned}$$

Continuing as before we get the result. □

Lemma 9 Under Assumption C we have the following estimates, for $i = 1, 2$ and $t \in [t_k, t_{k+1}]$,

$$\begin{aligned}\mathbb{E}|h_i(t) - y_i(t)|^2 &\leq C\Delta \\ \mathbb{E}|y_i(t) - y_i(t_k)|^2 &\leq C\Delta, \\ \mathbb{E}|h_i(t) - y_i(t_k)|^2 &\leq C\Delta, \\ \mathbb{E}|h_i(t)|^2 &\leq C\Delta.\end{aligned}$$

Proof. Using the moment bounds of Lemma 8 we easily get the result. \square

Lemma 10 Under Assumption B, we have the following estimates,

$$\mathbb{E}y_1(t)(\text{sgn}(z_1(t)) - 1)^2 \leq C\Delta^{\frac{1}{2}-\varepsilon}, \quad \mathbb{E}y_2(t)(\text{sgn}(z_2(t)) - 1)^2 \leq C\Delta^{\frac{1}{2}-\varepsilon}$$

for any $\varepsilon > 0$.

Proof. We begin with, when $t \in [t_k, t_{k+1}]$,

$$\begin{aligned}\mathbb{E}y_1(t)(\text{sgn}(z_1(t)) - 1)^2 &= 4\mathbb{E}y_1(t)\mathbb{I}_{\{z_1(t) \leq 0\}} \leq 4\mathbb{E}|y_1(t) - y_1(t_k)| + 4\mathbb{E}y_1(t_k)\mathbb{I}_{\{z_1(t) \leq 0\}} \\ &\leq C\Delta + 4\mathbb{E}y_1(t_k)\mathbb{I}_{\{z_1(t) \leq 0\}}\mathbb{I}_{\{y_1(t_k) \leq \Delta^{1/2-\varepsilon}\}} + 4\mathbb{E}y_1(t_k)\mathbb{I}_{\{z_1(t) \leq 0\}}\mathbb{I}_{\{y_1(t_k) > \Delta^{1/2-\varepsilon}\}} \\ &\leq C\Delta^{1/2-\varepsilon} + 4\mathbb{E}y_1(t_k)\mathbb{I}_{\{\{z_1(t) \leq 0\} \cap \{y_1(t_k) > \Delta^{1/2-\varepsilon}\}\}}\end{aligned}$$

We have used Lemma 9 to obtain the second inequality, estimating the term $\mathbb{E}|y_1(t) - y_1(t_k)|$. But

$$\begin{aligned}&\{z_1(t) \leq 0\} \cap \{y_1(t_k) > \Delta^{1/2-\varepsilon}\} \\ &= \left\{ W_t^1 - W_{t_k}^1 \leq -\frac{2}{\sigma_1} \sqrt{y_1(t_k)(1 - \lambda_{11}\Delta) + \Delta\lambda_{12}y_2(t_k) + \Delta(k - \frac{\sigma_1^2}{4})} \right\} \cap \{y_1(t_k) > \Delta^{1/2-\varepsilon}\} \\ &\subseteq \left\{ W_t^1 - W_{t_k}^1 \leq -\frac{2\sqrt{1 - \lambda_{11}\Delta}}{\sigma_1} \sqrt{\Delta^{1/2-\varepsilon}} \right\}.\end{aligned}$$

Since the increment $W_t^1 - W_{t_k}^1$ is normally distributed with mean zero and variance $t - t_k$ we have that

$$\mathbb{P}\left(\{z_1(t) \leq 0\} \cap \{y_1(t_k) > \Delta^{1/2-\varepsilon}\}\right) \leq C \frac{\sqrt{t - t_k}}{\sqrt{2\pi(t - t_k)}} \int_{\frac{2\sqrt{\Delta^{1/2-\varepsilon}}}{\sqrt{t - t_k}}}^{\infty} e^{-y^2/2} dy \leq \frac{C\Delta^\varepsilon}{e^{C/\Delta^\varepsilon}}.$$

We have used the inequality of problem 9.22, p.112 of [16] to obtain the last inequality. Now we have, using the moment bounds for the numerical solution,

$$\mathbb{E}y_1(t_k)\mathbb{I}_{\{\{z_1(t) \leq 0\} \cap \{y_1(t_k) > \Delta^{1/2-\varepsilon}\}\}} \leq C\mathbb{P}\left(\{z_1(t) \leq 0\} \cap \{y_1(t_k) > \Delta^{1/2-\varepsilon}\}\right)$$

Noting that $\frac{\Delta^\varepsilon}{e^{1/\Delta^\varepsilon}} \rightarrow 0$ faster than any power of Δ we have that

$$\mathbb{E}y_1(t)(\text{sgn}(z_1(t)) - 1)^2 \leq C\Delta^{\frac{1}{2}-\varepsilon}.$$

The same holds for $y_2(t)$. \square

Because h_1, h_2 are essential the same as in the previous section, we can use the same arguments as in Theorem 3 and Proposition 4 together with Lemma 10 to get the following results.

Theorem 4 If Assumption C holds then

$$\mathbb{E}(|x_1(t) - y_1(t)|^2 + |x_2(t) - y_2(t)|^2) \leq C \frac{1}{\sqrt{\ln n}}$$

for any $t \in [0, T]$.

Proposition 5 Suppose that Assumptions B and C hold. Then, if $\frac{L_1}{4(\sigma_1^2 + \sigma_2^2)} \geq 1$ and $\frac{L_2}{4(\sigma_1^2 + \sigma_2^2)} \geq 1$, the following rate of convergence holds,

$$\mathbb{E}|x_1(t) - y_1(t)|^2 + |x_2(t) - y_2(t)|^2 \leq C\Delta^{1/2-\varepsilon}.$$

for every $\varepsilon > 0$. That is the order of convergence is at least $1/4 - \varepsilon$.

Conclusion We have proposed two explicit and positivity preserving numerical schemes for the two factor CIR model. The first one is based on the exact simulation of the CIR process for a specific set of parameters. The advantage of the second method is that one need less calculations in each step comparing with the first method. However, extended numerical experiments has to be done to compare them. Let us mention that both the results hold for the case of one equation choosing for example $\lambda_{12} = 0$. Finally, the above results can be easily extended for the multi-factor case.

In [22] one can find a different use of the above model. If one considers a more complicated model than the above, for example,

$$\begin{aligned} x_1(t) &= x_1(0) + \int_0^t (k - \lambda_{11}x_1(s) + \lambda_{12}x_2(s))ds + \int_0^t \sigma_1 \sqrt{x_1(s)x_2(s)}dW_s^1, \\ x_2(t) &= x_2(0) + \int_0^t (l - \lambda_{21}x_2(s) + \lambda_{22}x_1(s))ds + \int_0^t \sigma_2 \sqrt{x_1(s)x_2(s)}dW_s^2 \end{aligned}$$

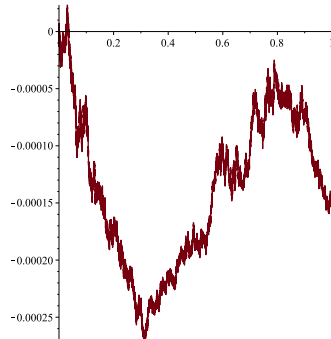
then it is not obvious how our first numerical scheme based on exact simulation can be applied here. Considering the second method one can propose the following numerical scheme,

$$\begin{aligned} y_1(t_{k+1}) &= \left(\frac{\sigma_1 \sqrt{y_2(t_k)}}{2} (W_{t_{k+1}}^1 - W_{t_k}^1) + \sqrt{y_1(t_k)(1 - \lambda_{11}\Delta) + \Delta\lambda_{12}y_2(t_k) + \Delta(k - \frac{\sigma_1^2 y_2(t_k)}{4})} \right)^2, \\ y_2(t_{k+1}) &= \left(\frac{\sigma_2 \sqrt{y_1(t_k)}}{2} (W_{t_{k+1}}^2 - W_{t_k}^2) + \sqrt{y_2(t_k)(1 - \lambda_{21}\Delta) + \Delta\lambda_{22}y_1(t_k) + \Delta(l - \frac{\sigma_2^2 y_1(t_k)}{4})} \right)^2. \end{aligned}$$

With the same analysis and with a minor modification on the hypotheses, one can prove that this scheme converges strongly to the true solution but without some rate, i.e. a similar result as Theorem 3.

As a minimal computer experiment we give below the difference between the numerical scheme (2) for $a = 1$ and the scheme proposed in [1] just to see that these methods are close. More complicated computer experiments has to be done in order to detect the actual order of convergence and other advantages or disadvantages of this method compared with that of [1].

Figure 1: $x_0 = 4$, $\Delta = 10^{-4}$, $k = 2$, $l = 1$, $s = 1$ $T = 1$.



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